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SHORT COMMUNICATION

BOUNDARY-FITTED CO-ORDINATES FOR THE ASYMPTOTIC FRICTION FACTORS AND NUSSELT NUMBERS OF LAMINAR FLOWS INSIDE STRAIGHT DUCTS WITH IRREGULAR, SINGLY CONNECTED CROSS-SECTIONS

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ABSTRACT

This paper presents a simple computational procedure for the laminar, fully established velocity and temperature in straight ducts with irregular, singly connected cross-sections by virtue of a control volume discretization of the momentum and energy equations in boundary-fitted co-ordinate systems. The combined procedure has been applied to a large group of ducts whose cross-sections possess different levels of difficulty. The numerical predictions for the pressure drop (friction factor) and the convective heat transfer coefficient (Nusselt number) have been reported for a sub-class of ducts with curved sides and sharp corners utilizing various grid sizes.

KEY WORDS Boundary-fitted co-ordinate systems Temperature Velocity

NOMENCLATURE

A_c = Cross sectional area of duct
c_p = Specific heat
D_h = Equivalent hydraulic diameter, $4 A_c/P$
f = Friction factor, equation (7)
h = Convective heat transfer coefficient
k = Thermal conductivity
Nu_{Dh} = Nusselt number, equation (8)
p = Perimeter of duct
dp/dz = Axial pressure gradient
Re_{Dh} = Reynolds number, $\rho \bar{w} D_h/\mu$
T = Temperature
w = Axial velocity
$\bar{w}$ = Mean axial velocity
x,y = Transverse co-ordinates
z = Axial co-ordinates

Greek letters
μ = Dynamic viscosity
ξ, η = Transformed co-ordinates
ρ = Density

Subscripts
o = Refers to inlet
w = Refers to wall

INTRODUCTION

Channels of complex, singly connected, cross sections are extensively employed in heat and mass exchange devices such as passages of turbomachinery, ducts of compact heat exchangers, and sub-channels of nuclear reactors. An inherent feature of fluid flows inside straight ducts of irregular cross section is that the shape of the cross plane may cause a marked distortion of the

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velocity and temperature profiles. Consequently, this distortion may enhance or lessen the corresponding changes in pressure and heat transfer.

Interestingly, the common denominator in the calculation procedure of friction factors for basic ducts is its simplicity. In contrast; computation of the Nusselt number is not as straightforward. For the circular tube with a constant wall temperature, Bhatti[1] examined an infinite series solution for the developed temperature profile and was able to predict the exact value for the Nusselt number, $Nu_D = 3.657$. In principle, his technique could have been extended to other regular ducts whose sides conform to either Cartesian or cylindrical co-ordinate systems, e.g. parallel-plate channels and annular passages. However, it must be anticipated that the analysis of ducts with irregular cross sections whose sides cannot be accommodated into standard co-ordinate systems is more elaborate.

Historically, it has been recognized that numerical solutions of partial differential equations in regions of arbitrarily shaped boundaries are difficult. These solutions are usually generated by finite-difference techniques or finite element methods (see Ames[2]).

In the present note an attempt is made to analyse numerically the forced convection patterns of hydrodynamically and thermally developed flows inside straight ducts with arbitrarily-shaped, but singly connected cross sections. As observed by Mills[3], one important classification of heat exchangers is the so-called single-stream heat exchanger in which the temperature of only one stream varies, while the temperatures of the other streams remain unaltered. Examples of these heat exchangers include different types of evaporators and condensers.

The computational procedure that is outlined in the following sections is based on a suitable combination of boundary-fitted co-ordinates and the finite-volume method. Consequently, once the velocity and temperature profiles have been accurately calculated, the pressure gradients and the convective heat transfer can be numerically determined.

CONSERVATION EQUATIONS

Attention is focused on viscous fluids flowing inside straight ducts having irregular, singly connected cross sections and isothermal, thin-walls. Assuming laminar, hydrodynamic developed and thermally developing conditions, the momentum and energy conservation equations for such fluids having constant thermophysical properties are conveniently modelled in Cartesian co-ordinates as follows:

$$0 = -\frac{dp}{dz} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \tag{1}$$

$$\rho c_p w \frac{\partial T}{\partial z} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{2}$$

where $w = w(x,y)$ designates the axially-invariant velocity profile. Note that the terms involving axial diffusion of both momentum and heat are not included in these equations for simplicity.

$$w = 0,\ T = T_w, \text{ at the duct surface} \tag{3}$$

$$T = T_0, \text{ at the entrance, } z = 0. \tag{4}$$

IMPLEMENTATION OF BOUNDARY FITTED CO-ORDINATES

The conservation equations, in conjunction with the boundary conditions, are transformed from a physical co-ordinate system (x,y) into a computational co-ordinate system (ξ,η) by adopting suitable boundary fitted co-ordinates. The details of the transformation relations are omitted for brevity, but the interested reader may consult Thompson *et al.*[4]. Consequently, the converted momentum and energy equations are rewritten as

$$0 = -\frac{dp}{dz} + \frac{\mu}{J}\left[\frac{\partial}{\partial\xi}\left(\frac{c_1}{J}\frac{\partial w}{\partial\xi} - \frac{c_2}{J}\frac{\partial w}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(\frac{c_3}{J}\frac{\partial w}{\partial\eta} - \frac{c_2}{J}\frac{\partial w}{\partial\xi}\right)\right] \quad (5)$$

$$\rho c_p w\frac{\partial T}{\partial z} = \frac{k}{J}\left[\frac{\partial}{\partial\xi}\left(\frac{c_1}{J}\frac{\partial T}{\partial\xi} - \frac{c_2}{J}\frac{\partial T}{\partial\eta}\right) + \frac{\partial}{\partial\eta}\left(c_3\frac{\partial T}{\partial\eta} - c_2\frac{\partial T}{\partial\xi}\right)\right] \quad (6)$$

respectively, where c_1, c_2, and c_3 identify the transformation coefficients, whereas J denotes the Jacobian matrix.

NUMERICAL COMPUTATIONAL PROCEDURE

The calculation procedure for the two conservation equations in the transformed computational domain is based on the finite-volume discretization of Patankar[5]. The domain is divided into contiguous control volumes and to increase accuracy; these control volumes account for nine nodes, instead of the standard five nodes. Some relevant concepts and details previously used in solving laminar, fully developed, non-isothermal flow with heat transfer in regular ducts have been extended to the situations envisaged in the present work.

The procedure for solving the two systems of algebraic equations for the field variables: velocity and the temperature was the standard line-by-line method explained in reference 5. In addition, the block correction procedure of Settari and Aziz[6] was incorporated to enhance convergence.

Normally, for any duct regardless of the cross sectional shape, the pressure drop is calculated by the friction factor:

$$fRe_{D_h} = \frac{D_h^2}{2\overline{w}\mu}\frac{dp}{dz} \quad (7)$$

and the heat transfer is determined by the Nusselt number:

$$Nu_{D_h} = \frac{h\,D_h}{k} \quad (8)$$

respectively. Both quantities, being of global character, relied on the equivalent hydraulic diameter, D_h.

The accuracy of the computer code has been validated by using a known standard shape. Thus, the circular tube was chosen, where the fully developed values of friction factor and Nusselt number are $fRe_D = 64$ and $Nu_D = 3.657$, respectively.

Directing the attention to the sensitivity of the computed numerical results, various grid sizes have been carefully tested in order to guarantee the highest orthogonality of the grid for this simple shape. At this point, it is important to underline that Thompson *et al.*[4] have concluded that moderate departures from orthogonality in the grids do not have repercussions on errors in the calculation of global parameters. Their findings were based on actual calculations.

The computations were carried out with three grids which at first were considered adequate: a 10 × 10 coarse grid; a 20 × 20 moderate grid; and a 40 × 40 fine grid. As far as the accuracy is concerned, each of the velocity and temperature fields was deemed to converge when the maximum normalized residuals for both reached values between 10^{-4} and 10^{-6}. The moderate grid with 20 × 20 points produced results for fRE_D and NU_D that were within a 1 per cent error. Moreover, it was found that as the grid becomes finer NU_D converges faster than fRE_D to their respective real values. The 10 × 10 coarse grid supplied a value of $fRe_D = 62.02$ which lies within –3.09 per cent of the true solution exhibiting a maximum deviation and then departing very rapidly to 63.48 with an accompanying – 0.81 per cent error for a moderate grid of 20 × 20 nodes. In contrast, the same 10 × 10 grid furnished values of $Nu_D = 3.69$, which has a diminute error of 0.81 per cent. These numbers become 3.67 and 0.32 per cent respectively for a 20 × 20 grid. Finally, for a fine 40 × 40 grid, the deviations between the calculated and real values of both $fRe_D = 64$ and $Nu_D = 3.658$ are imperceptible. This comparison points out the manner in which the results converge towards the actual solution after systematically refining the grid.

PRESENTATION OF RESULTS

This section pertains to the assessment of the hydrodynamics and heat transfer characteristics highlighting the suitability of the combined numerical procedure for a general class of non-circular passages. From an engineering standpoint, the relevant quantities for analysis and design of these ducts are: the pressure drop; and the heat transfer rate. For generality, both quantities are expressed by the hydraulic diameter. *Tables 1* and *2* have been prepared to illustrate a comparison of $f\mathrm{Re}_{Dh}$ and Nu_{Dh} for a highly demanding duct configuration whose geometries exhibited curved sides and sharp corners, e.g. moon ducts (see *Figures 1* and *2*). The moon duct is a duct formed by two circular arcs which are related by the function cos $\phi = b/(2a)$; the radius of the outer and inner arcs are a and b, respectively. The same grid sizes employed before for the circular tube have been implemented for this geometry.

The patterns already observed for a circular tube are repeated for a moon duct with an arc ratio of $2\phi = 60°$, namely the 20 × 20 moderate grid provided satisfactory results within 1 per cent margin of error. Analogously, for this grid *fRe*Dh converged much faster than the corresponding Nu_{Dh} to their respective true values. On refinement of the grid to 40 × 40 nodes, the opposite trend was observed, namely Nu_{Dh} converges faster than fRe_{Dh} to their respective real values. Conversely, for a more intricate shape, like the moon duct with an arc ratio of $2\gamma = 120°$, the 20 × 20 moderate grid does not suffice to accommodate the stiff velocity and temperature distortions. This grid produces global results for fRe_{Dh} that stay within 2 per cent, whereas for Nu_{Dh} the errors are contained within 1 per cent error. Nevertheless, the stringent criterion of 1 per cent can be perfectly satisfied with a finer grid, such as a 40 × 40 grid.

From a conceptual standpoint, other boundary conditions do not need to be tested because the Nusselt numbers for isothermal wall conditions, $Nu_{Dh,T}$, can be safely used to estimate the other

Table 1 Friction factors and Nusselt numbers for a moon shaped duct with $2\phi = 60°$

Grid	fRe_{Dh}	Percentage error	Nu_{Dh}	Percentage error
10 × 10	59.93	−2.79	2.57	
20 × 20	61.23	−0.68	2.63	
40 × 40	61.55	−0.16	2.64	
Shah[7]	61.65		–	

Table 2 Friction factors and Nusselt numbers for a moon shaped duct with $2\phi = 120°$

Grid	fRe_{Dh}	Percentage error	Nu_{Dh}	Percentage error
10 × 10	55.49	−7.69	2.79	
20 × 20	58.87	−2.06	2.93	
40 × 40	59.80	−0.52	2.96	
Shah[7]	60.11		–	

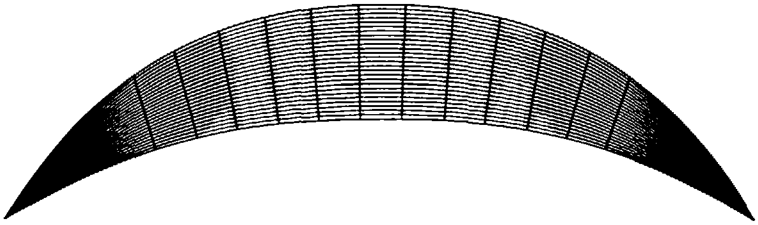

Figure 1 Moon shaped duct with $\phi = 60°$ (20 × 20 grid)

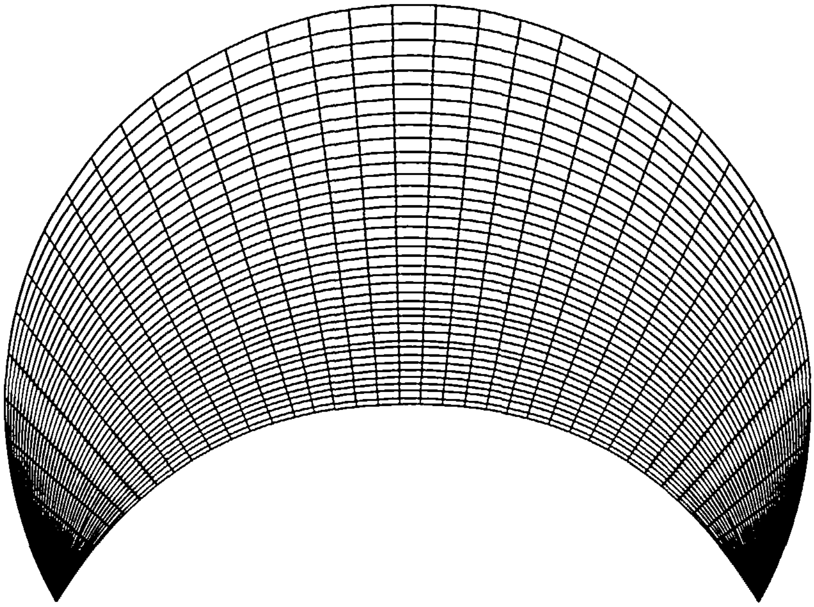

Figure 2 Moon shaped duct with $\phi = 120°$ (40 × 40 grid)

Nusselt numbers for isoflux conditions, $Nu_{Dh,T}$. Under normal circumstances, the latter is slightly higher than the former, deviating by a small percentage only. A compilation of the $Nu_{Dh,H}/Nu_{Dh,T}$ ratios for a wide variety of tubes with singly connected, cross sectional shapes is gathered in Table 138 of Reference 7, providing evidence that the ratios vary smoothly from 1.09 to 1.26.

To save journal space, it was decided to omit the heat and fluid flow results for a collection of equally important duct configurations with singly-connected regions that are delineated in Reference 7.

CONCLUDING REMARKS

Verification of the numerical predictions has been done on a straight circular tube (test case) prior to the inclusion of irregular ducts in the boundary-fitted co-ordinate system. After a series of numerical experiments, grid independence has been achieved on 20 × 20 grids for the majority of the ducts tested. Furthermore, ducts with intricate shapes required a much finer grid, like 40 × 40. It may reasonably be concluded that the finite volume discretization in conjunction with a boundary-fitted co-ordinate system is a versatile and effective route for the study of fully developed flow and temperature in channels with singly connected complex geometries. Actually, variation of the grid sizes depends primarily on the intricacy of the cross-section and the precision level needed.

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